UDC 539.3

# ON A PLANE ELASTICITY THEORY PROBLEM WITH MIXED BOUNDARY CONDITIONS

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Making use of the method presented in [1], we solve the plane elasticity theory problem of the plane with a rectilinear cut, having mixed boundary conditions at its edges.

**1.** Assume that the domain $S'$, occupied by a homogeneous, elastic and isotropic medium, is represented by the plane of the complex variable $z = x + iy$, cut along the segment $L = ab$ of the real axis $Ox$. We take as the positive direction along $L$ the positive direction of the $Ox$-axis and we will assign to the boundary values of the considered functions at the left and right-hand side of $L$ the index plus and minus, respectively.

The problem consists in the determination of the state of stress of the elastic medium uder the following boundary conditions:

$$v^+ = f_1(t), \quad v^- = f_2(t) \quad \text{on} \quad L \tag{1.1}$$

$$X_y^+ = kY_y^+, \quad X_y^- = kY_y^- \quad \text{on } L \tag{1.2}$$

where $k = \text{const} > 0$, while $f_1(t)$ and $f_2(t)$ $(t \in L)$ are given functions whose values together with their derivatives $f_1'(t)$ and $f_2'(t)$, are small quantities of the order of the admissible displacements. We consider that $f_1'(t)$ and $f_2'(t)$ satisfy the Hölder condition on $L$. Here and in the sequel we make use of the definitions and notations adopted in [1]. We also assume that the following conditions hold:

$$v^+(a) = v^-(a), \quad v^+(b) = v^-(b) \tag{1.3}$$

and we specify [1]

$$\Gamma = B + iC, \; \Gamma' = B' + iC', \; C = 0 \tag{1.4}$$

so that at infinity the stresses are bounded and the rotation is equal to zero. We further assume that the resultant vector $(X, Y)$ of the external forces applied to both edges of the cut $L$ is satisfied, i.e. $Y = -\int\limits_L [p(t) - q(t)]\, dt, \qquad X = kY$

$$p(t) = Y_y^+, \; q(t) = Y_y^-, \; (t \in L)$$

It is known [1] that the elastic equilibrium of $S'$ is defined in terms of two functions $\Phi(z)$ and $\Omega(z)$, holomorphic in $S'$, including the point at infinity, which for large $|z|$ have the form (see [1] Sect. 120):

$$\Phi(z) = \Gamma - \frac{X + iY}{2\pi(\varkappa + 1)} \frac{1}{z} + O\left(\frac{1}{z^2}\right) \qquad (1 < \varkappa < 3)$$

$$\Omega(z) = \bar{\Gamma} + \bar{\Gamma}' + \frac{\varkappa(X + iY)}{2\pi(\varkappa + 1)} \frac{1}{z} + O\left(\frac{1}{z^2}\right) \tag{1.5}$$

Here $\varkappa$ is a constant which characterizes the elastic properties of the medium. We consider that in the neighborhood of each of the extremities $a$ and $b$ the following estimates ($c$ is the corresponding extremity)

$$|\Phi(z)|<\frac{A}{|z-c|^{\alpha}},\ |\Omega(z)|<\frac{A}{|z-c|^{\alpha}} \qquad (A>0,\ 0<\alpha<1)$$

hold. In addition, we consider that at all points $t\in L$ which do not coincide with the extremities

$$\lim_{z\to t} y\Phi'(z)=0 \qquad (z=x+iy) \tag{1.6}$$

Making use of the formulas (see [1], Sect. 120)

$$Y_y-iX_y=\Phi(z)+\Omega(\bar{z})+(z-\bar{z})\overline{\Phi'(z)}$$

$$2\mu\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)=\varkappa\Phi(z)-\Omega(\bar{z})-(z-\bar{z})\overline{\Phi'(z)}$$

and taking into account (1.6), we transform the boundary conditions (1.1), (1.2) into the following form ($\mu$ is Lamé constant):

$$\varkappa[\Phi^+(t)-\overline{\Phi^+(t)}]-[\Omega^-(t)-\overline{\Omega^-(t)}]=4\mu i f_1'(t) \quad \text{on } L$$

$$\varkappa[\Phi^-(t)-\overline{\Phi^-(t)}]-[\Omega^+(t)-\overline{\Omega^+(t)}]=4\mu i f_2'(t) \quad \text{on } L \tag{1.7}$$

$$[\Phi(t)-\Omega(t)]^+-[\Phi(t)-\Omega(t)]^-=(1-ik)[p(t)-q(t)] \quad \text{on } L$$

$$[\Phi(t)+\Omega(t)]^++[\Phi(t)+\Omega(t)]^-=(1-ik)[p(t)+q(t)] \quad \text{on } L \tag{1.8}$$

Thus, the formulated problem has been reduced to the determination of the functions $\Phi(z)$ and $\Omega(z)$ from the conditions (1.7), (1.8), where $p(t)$ and $q(t)$ are unknown real functions.

Since $\Phi(\infty)-\Omega(\infty)=-\overline{\Gamma}'$, the general solution of the first of the boundary value problems (1.8) can be expressed by the formula [1]

$$\Phi(z)-\Omega(z)=\frac{1-ik}{2\pi i}\int_L\frac{p(\tau)-q(\tau)}{\tau-z}\,d\tau-\overline{\Gamma}',\quad -\overline{\Gamma}'=C_0 \tag{1.9}$$

Then, setting

$$X(z)=\sqrt{(z-a)(b-z)},\qquad \lim_{z\to\infty} z^{-1}X(z)=-i$$

for the general solution of the second of the boundary value problems (1.8), we obtain [1]

$$\Phi(z)+\Omega(z)=\frac{1-ik}{2\pi iX(z)}\int_L\frac{X^+(\tau)[p(\tau)+q(\tau)]}{\tau-z}\,d\tau+\frac{C_1z+C_2}{X(z)} \tag{1.10}$$

Here by $X^+(t)=\sqrt{(t-a)(b-t)}$ we mean the values taken by $X(z)$ on the upper edge of the cut $L$, while $C_1$ and $C_2$ are arbitrary complex constants subject to determination. Taking into account that for large $|z|$

$$\frac{1}{X(z)}=\frac{i}{z}+i\frac{a+b}{2}\frac{1}{z^2}+\ldots$$

from (1.10) we obtain

$$\Phi(z)+\Omega(z)=iC_1+i\left(C_2+\frac{a+b}{2}C_1\right)\frac{1}{z}+O\left(\frac{1}{z^2}\right)$$

On the other hand, from (1.5) we have

$$\Phi(z)+\Omega(z)=2B+\overline{\Gamma}'+\frac{(\varkappa-1)(X+iY)}{2\pi(\varkappa+1)}\frac{1}{z}+O\left(\frac{1}{z^2}\right)$$

From the last two equalities we obtain for the constants $C_1$ and $C_2$

$$C_1 = -i\,(2B + \overline{\Gamma}')$$

$$C_2 = \frac{(\varkappa - 1)(1 - ik)\,Y}{2\pi\,(\varkappa + 1)} + i\frac{a + b}{2}(2B + \overline{\Gamma}') \tag{1.11}$$

In the sequel we assume that $C_1$ and $C_2$ are known. From (1.9) and (1.10) we determine $\Phi(z)$ and $\Omega(z)$. Making use of the Sokhotskii-Plemelj formulas [2, 3] and taking into account that $X^-(t) = -X^+(t)$ on $L$, we obtain

$$2\Phi^{\pm}(t) = (1 - ik)\left\{\pm\frac{p(t) - q(t)}{2} + \frac{p(t) + q(t)}{2} + \frac{1}{2\pi i}\int_L \frac{p(\tau) - q(\tau)}{\tau - t}\,d\tau \pm \right.$$
$$\left.\frac{1}{2\pi i X^+(t)}\int_L \frac{X^+(\tau)[p(\tau) + q(\tau)]}{\tau - t}\,d\tau\right\} + C_0 \pm \frac{C_1 t + C_2}{X^+(t)} \tag{1.12}$$

$$2\Omega^{\pm}(t) = (1 - ik)\left\{\mp\frac{p(t) - q(t)}{2} + \frac{p(t) + q(t)}{2} - \frac{1}{2\pi i}\int_L \frac{p(\tau) - q(\tau)}{\tau - t}\,d\tau \pm \right.$$
$$\left.\frac{1}{2\pi i X^+(t)}\int_L \frac{X^+(\tau)[p(\tau) + q(\tau)]}{\tau - t}\,d\tau\right\} - C_0 \pm \frac{C_1 t + C_2}{X^+(t)}$$

Substituting $\Phi^{\pm}(t)$ and $\Omega^{\pm}(t)$ from (1.12) into the equalities (1.7), after some simple transformations we obtain for the unknown functions $p(t)$ and $q(t)$ the following system of singular integral equations:

$$(1 - \varkappa)\,k\,[p(t) + q(t)] - \frac{\varkappa + 1}{\pi}\int_L \frac{p(\tau) - q(\tau)}{\tau - t}\,d\tau = 4\mu\,[f_1'(t) + f_2'(t)] - 2(\varkappa + 1)\,b_0$$

$$(1 - \varkappa)\,k\,[p(t) - q(t)] - \frac{\varkappa + 1}{\pi X(t)}\int_L \frac{X(\tau)[p(\tau) + q(\tau)]}{\tau - t}\,d\tau =$$
$$4\mu\,[f_1'(t) - f_2'(t)] - 2(\varkappa + 1)\frac{b_1 t + b_2}{X(t)} \tag{1.13}$$

Here by $X(t)$ we have denoted $X^+(t)$, while $b_0$, $b_1$, $b_2$ are determined from the equalities

$$C_0 = a_0 + ib_0,\quad C_1 = a_1 + ib_1,\quad C_2 = a_2 + ib_2$$

Making use of the methods given in [4], we can prove on the basis of the assumptions introduced that the system (1.13) has a unique solution given by the following expression:

$$p(t) + q(t) = -\frac{4\mu k(\varkappa - 1)[f_1'(t) + f_2'(t)]}{(\varkappa + 1)^2 + k^2(\varkappa - 1)^2} + \frac{4\mu(\varkappa + 1)}{\pi[(\varkappa + 1)^2 + k^2(\varkappa - 1)^2]} \times$$
$$\int_L \frac{f_1'(\tau) - f_2'(\tau)}{\tau - t}\,d\tau + \frac{2k(\varkappa^2 - 1)\,C' + 2(\varkappa + 1)^2(2B + B')}{(\varkappa + 1)^2 + k^2(\varkappa - 1)^2} \tag{1.14}$$

$$p(t) - q(t) = -\frac{4\mu k(\varkappa - 1)[f_1'(t) - f_2'(t)]}{(\varkappa + 1)^2 + k^2(\varkappa - 1)^2} + \frac{4\mu(\varkappa + 1)}{\pi[(\varkappa + 1)^2 + k^2(\varkappa - 1)^2]\,X(t)} \times$$
$$\int_L \frac{X(\tau)[f_1'(\tau) + f_2'(\tau)]}{\tau - t}\,d\tau + \frac{[(\varkappa + 1)^2 C' - k(\varkappa^2 - 1)(2B + B')](2t - a - b)}{[(\varkappa + 1)^2 + k^2(\varkappa - 1)^2]\,X(t)} - \frac{Y}{\pi X(t)}$$

It should be noted that the result obtained holds also in the case $k = 0$.

Since $p(t) + q(t)$ and $p(t) - q(t)$ are already known, from (1.9) and (1.10) we obtain $\Phi(z) + \Omega(z)$ and $\Phi(z) - \Omega(z)$, consequently, also $\Phi(z)$ and $\Omega(z)$. This solves the formulated problem.

**2.** We consider an example. Assume that a symmetric cut $L \equiv [-l, l]$ is given on which

$$v^+ = \frac{1}{2R}(l^2 - t^2), \qquad v^- = \frac{1}{2R}(t^2 - l^2)$$

where the constant $R > 0$ is sufficiently large. We assume that on both edges of $L$ the relations (1.2) are satisfied. We also assume that the stresses and the rotation vanish at infinity and that the resultant vector of the external forces acting on both edges of the cut $L$ is equal to zero.

Under these assumptions, taking into account (1.4) and (1.11), we set $B = B' = C' = 0$, from where by virtue of the formulas (1.14) we obtain

$$p(t) + q(t) = -\frac{8\mu(\varkappa+1)}{\pi R[(\varkappa+1)^2 + k^2(\varkappa-1)^2]}\left(2l + t\ln\frac{l-t}{l+t}\right)$$

$$p(t) - q(t) = \frac{8\mu k(\varkappa-1)\, t}{R[(\varkappa+1)^2 + k^2(\varkappa-1)^2]}$$

Substituting the expressions for $p(t) + q(t)$ and $p(t) - q(t)$ into (1.9) and (1.10), respectively, we obtain for the functions $\Phi(z)$ and $\Omega(z)$

$$\Phi(z) = -\frac{2\mu(1-ik)}{\pi R[\varkappa + 1 - ik(\varkappa-1)]}\left(2l + z\ln\frac{z-l}{z+l}\right)$$

$$\Omega(z) = -\frac{2\mu(1-ik)}{\pi R[\varkappa + 1 + ik(\varkappa-1)]}\left(2l + z\ln\frac{z-l}{z+l}\right)$$

Here by $\ln[(z-l)/(z+l)]$ we mean the branch which is holomorphic in the plane cut along $L$ and which vanishes at infinity.

The obtained functions $\Phi(z)$ and $\Omega(z)$ determine the state of stress of the elastic medium which occupies the domain $S'$.

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